ON THE MAGNETOHYDRODYNAMIC HYPERSONIC FLOW PAST A WEDGE

(O MAGNITOGIDRODINAMICHESKOM GIPERZVUKOVOM OBTEKANII KLINA)

PMM Vol.27, No.1, 1963, pp. 55-60

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(Received September 18, 1962)

We consider the hypersonic flow past a wedge in which a magnetic field is energized, under the assumption that the region between the body and the shock wave is narrow. In [1], the solution was obtained for the case where the direction of the magnetic field is perpendicular to the surface of the body. Here we shall assume the angle between the magnetic field and the body to be arbitrary. In contrast to the solution in [1], in the present case, the pressure on the wedge differs from that calculated from the Newtonian formula. As a result, instead of the separation [1] (the velocity at some point on the surface vanishing), cavitation may occur under definite conditions, i.e. the pressure at some point may vanish in the presence of the magnetic field.

We find the locations of the separation and cavitation points as functions of the wedge angle and the angle of inclination of the magnetic field.

1. We shall use the equations of magnetohydrodynamics in orthogonal curvilinear coordinates x (length along the body) and y (distance along the normal to the body). The gas is assumed to be ideal, perfect, and finitely-conducting in the region behind the shock wave. The flow region between the body and the shock is assumed to be sufficiently narrow [2,3], i.e. $\varepsilon = (\kappa - 1)/(\kappa + 1) \ll 1$ (where κ is the adiabatic exponent), and we neglect in the equations quantities of order ε in comparison with unity. As a result, we obtain

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\frac{qu \sin^2 \alpha}{\rho}, \qquad \frac{\partial p}{\partial y} = qu \sin \alpha \cos \alpha + \frac{\rho u^2}{R}$$

$$u^2 + \frac{p}{\varepsilon\rho} = 1, \qquad \frac{\partial \rho u r^\nu}{\partial x} + \frac{\partial \rho v r^\nu}{\partial y} = 0, \qquad q = \frac{\sigma L H^2}{c^2 \rho_\infty U_\infty{}^2} \tag{1.1}$$

We use the following notation: u and v are the velocity components along the x- and y-axes respectively; α is the angle between the tangent to the body contour and the magnetic field vector $\mathbf{H}$; p is the pressure; ρ is the density; σ is the specific electrical conductivity (below, we assume σ = const in the shock layer, and $\sigma = 0$ in the unperturbed flow); c is the velocity of light in vacuum; ρ_∞ and U_∞ are respectively the density and velocity of the unperturbed flow; R is the radius of curvature of the contour; L is a characteristic length; $r = r(x)$ is the distance from a point on the body contour to the axis of symmetry; and $\nu = 0$, 1 respectively, for plane and axisymmetric flows.

Equation (1.1) is reduced to dimensionless form, with U_∞, ρ_∞ and $\rho_\infty U_\infty{}^2$ as the characteristic quantities for the velocity, density, and pressure respectively, while x and y are expressed relative to L. We assume that the nondimensional magnetic field intensity parameter q is of order ε^{-1}.

As was shown in [1], the induced magnetic field may be neglected in comparison with the given field for magnetic Reynolds' numbers $R_m \lesssim 1$. This permits the approximation of separating the problem of calculating the fluid dynamic field from equations (1.1) with a given magnetic field, from the problem of calculating the induced magnetic field, which we ignore here. Moreover, to the assumed accuracy, the magnetic field may be considered constant across the shock layer, so that $\mathbf{H} = \mathbf{H}(x)$.

To the assumed accuracy, the boundary conditions may be written in the form (subscript 0 refers to quantities immediately behind the shock wave, and subscript 1 to quantities at the wall)

$$u_0 = \cos\theta, \quad \rho_0 = \varepsilon^{-1}, \quad p_0 = \sin^2\theta, \quad \theta_0 = \cos\theta\,(dy_0/dx - \varepsilon \tan\theta), \quad v_1 = 0 \tag{1.2}$$

$(y = y_0(x)$ — is the equation of the shock wave)

Here $\theta = \theta(x)$ is the angle between the tangent to the body contour and the unperturbed free stream velocity. In [1], equations (1.1) were solved for the flows around a wedge and a cone (for the cone the solution was given in other coordinates) when $\alpha = \pi/2$. We note that equations (1.1) were also considered in [4], but no example of a solution was given there. As follows from the second equation in (1.1), the presence of the magnetic field directed at an arbitrary angle $\alpha \neq \pi/2$ to the body contour is apparently equivalent to some fictitious centrifugal force, which, depending on the sign of sin 2α, is directed either toward the body or away from it; in other words, it reduces to an increase or decrease in pressure in agreement with the value calculated from Busemann's formula [2,3].

We transform equation (1.1) to the independent variables of Crocco x and ψ (the stream function), where ψ is defined by the expressions

$$\frac{\partial\psi}{\partial x} = \rho v r^{\nu}, \qquad \frac{\partial\psi}{\partial y} = -\rho u r^{\nu} \tag{1.3}$$

As a result, we get*

$$\frac{\partial u}{\partial x} + \frac{q\sin^2\alpha}{\rho} = 0, \qquad -\rho r^{\nu}\frac{\partial p}{\partial\psi} = q\sin\alpha\cos\alpha + \frac{\rho u}{R}$$
$$u^2 + \frac{p}{\varepsilon\rho} = 1, \qquad \frac{\partial}{\partial x}\frac{1}{\rho u r^{\nu}} + \frac{\partial}{\partial\psi}\frac{v}{u} = 0 \tag{1.4}$$

The equation of the shock wave in these coordinates become

$$\psi_0(x) = -\int_0^x r^{\nu}\sin\theta\,dx - \int_0^{y_0} r^{\nu}\cos\theta\,dy \tag{1.5}$$

The equation of the body surface is $\psi = 0$. On the line $\psi = \psi_0(x)$, conditions (1.2) obtain, the last equation of which becomes in the new variables

$$\frac{\partial\psi_0}{dx} = -r^{\nu}[v_0 + (1+\varepsilon)\sin\theta] \tag{1.6}$$

while the other conditions remain unchanged.

The first term on the right-hand side of equation (1.5) is of order unity, while the second is of order ε. This permits solving the problem to the assumed accuracy by two successive approximations. First, we solve the Cauchy problem for the first three equations in (1.4) under the conditions

$$u_0 = \cos\theta, \quad p_0 = \sin^2\theta, \quad \rho_0 = \varepsilon^{-1} \quad \text{for } \psi_0(x) = -\int_0^x r^{\nu}\sin\theta\,dx \tag{1.7}$$

From this solution, when the magnetic field strength tends to zero ($q \to 0$), we obtain the well-known Busemann pressure formula [2,3]. Using

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* The transition from the first equation of (1.1) to the first equation of (1.4) results from an elementary identity of the transformation, and not from neglecting the term $v\partial u/\partial y$ in comparison with $u\partial u/\partial x$ in (1.1), as was stated in [5], where some incorrect critical remarks were made of [1].

the obtained values $\rho(x, \psi)$ and $u(x, \psi)$, we determine v from the last equation (1.4) and the boundary condition (no flow through the wall) by quadrature

$$v(x,\psi) = -u\int_0^{\psi} \frac{\partial}{\partial x}\left(\frac{1}{\rho u r^{\nu}}\right) d\psi \tag{1.8}$$

Setting in (1.8)

$$\psi_0 = -\int_0^{x} r^{\nu} \sin\theta \, dx$$

and substituting the obtained expression for v_0 into equation (1.6), we obtain an equation for the stream function on the shock wave in the next approximation, from which, using (1.5), we may determine the equation of the shock wave $y_0 = y_0(x)$.

2. Let us consider the solution to the problem of plane flow about the wedge, for the case where the magnetic field is independent of x (H = const, α = const).

In equations (1.4), we set $\nu = 0$, $R = \infty$; then the first two equations may be written in the following form, when the third equation is taken into account

$$\frac{\partial u}{\partial x} + \frac{q\varepsilon \sin^2\alpha\,(1-u^2)}{p} = 0, \qquad \frac{\partial p}{\partial \psi} = \cot\alpha\,\frac{\partial u}{\partial x} \tag{2.1}$$

with the boundary conditions

$$u_0 = \cos\theta, \quad p_0 = \sin^2\theta \quad \text{for } \psi_0 = -x\sin\theta \tag{2.2}$$

The hyperbolic system (2.1) with boundary conditions (2.2) admits an exact solution of the form

$$u = u(\eta), \qquad p = p(\eta) \qquad (\eta = \psi + x\sin\theta)$$

moreover, from the second equation of (2.1), we have

$$p = \frac{\sin\theta}{\sin\alpha}\left[u\cos\alpha - \cos(\theta+\alpha)\right] \tag{2.3}$$

Substituting expression (2.3) for p in the first equation of (2.1) and integrating the resulting equation for u, we obtain the solution

$$\left(\frac{1-u^2}{\sin^2\theta}\right)^{\cos\alpha}\left(\frac{1+u}{1-u}\tan^2\frac{\theta}{2}\right)^{\cos(\theta+\alpha)} = \exp\frac{2q\varepsilon\sin^3\alpha\,(\psi + x\sin\theta)}{\sin^2\theta} \tag{2.4}$$

Without loss of generality, we may consider in equations (2.3) and (2.4) $0 \leqslant \alpha \leqslant \pi$, since changing the sign of the magnetic field vector (i.e. changing the angle from α to $\alpha + \pi$) does not alter the solution. After determining u from (2.4), we find the pressure p from (2.3), and the density ρ from the third equation in (1.4). In order to find the distribution of u, p and ρ along the surface, it is only necessary to set $\psi = 0$ in the solution.

From equations (2.3) and (2.4), it follows that the velocity along the surface of the wedge always decreases with increasing x. The pressure drops along the surface for $0 < \alpha < \pi/2$, when the action of the magnetic field is equivalent to the centrifugal force in a flow about a convex body, and rises with increase in x for $\pi/2 < \alpha < \pi$, when the flow is equivalent to that about some concave body. When the angle $\theta + \alpha$ between the magnetic field and the direction of the unperturbed flow velocity is less than $\pi/2$, at a point on the body surface with abscissa $x = x_*$, where $u = \cos(\theta + \alpha)/\cos\alpha$, the pressure becomes zero; i.e. cavitation results from the presence of the magnetic force. The point where $p = 0$ is a singular point for equations (1.1); in some neighborhood of this singular point the solution loses its validity, since the basic assumption $\rho \sim \varepsilon^{-1}$ is violated there. It is clear that the solution (2.3), (2.4), cannot be extended into the region $x > x_*$.

For $\theta + \alpha > \pi/2$, at some point with abscissa $x = x_{**}$, the velocity u becomes zero, i.e. separation occurs [1]. The point $x = x_{**}$ is also a singular point of the solution, and the solution cannot be extended into the region $x > x_{**}$. Determining the cavitation and separation points from (2.3) and (2.4), we find

$$q\varepsilon x_* = \frac{\sin\theta}{2\sin^3\alpha}\ln\left\{\left[\frac{\sin(\theta + 2\alpha)}{\cos^2\alpha\sin\theta}\right]^{\cos\alpha}\left[\cot\frac{\theta}{2}\tan\left(\alpha + \frac{\theta}{2}\right)\right]^{-\cos(\theta+\alpha)}\right\} \qquad (2.5)$$

$(\theta + \alpha < \pi/2$, **solution valid in region** $x < x_*)$

$$q\varepsilon x_{**} = \frac{\sin\theta}{\sin^3\alpha}\ln\left[\frac{1}{(\sin\theta)^{\cos\alpha}}\left(\tan\frac{\theta}{2}\right)^{\cos(\theta+\alpha)}\right] \qquad (2.6)$$

$(\theta + \alpha > \pi/2$, **solution valid in region** $x < x_{**})$

In the case of cavitation $u > 0$ for $x < x_*$, and in the case of separation $p > 0$ for $x < x_{**}$. For $\theta + \alpha = \pi/2$, an intermediate case occurs, when the separation and cavitation points coincide

$$q\varepsilon x_* = q\varepsilon x_{**} = \frac{\tan^2\theta}{\cos\theta}\ln\frac{1}{\sin\theta} \qquad (\theta + \alpha = \pi/2) \qquad (2.7)$$

As an example, we give in Fig. 1 the quantities $q\varepsilon x_*$ and $q\varepsilon x_{**}$ as functions of the angle α (the inclination of the magnetic field) for $\theta = 45^\circ$. Both curves join smoothly for $\alpha = 45^\circ$, when (2.7) holds. It is

interesting to observe that for $\alpha = 71^{\circ}$, the quantity x_{**} has a minimum; i.e. in this case, the effect of a magnetic field of given intensity on

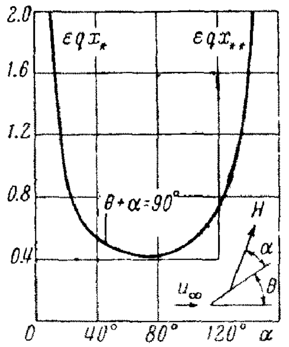

Fig. 1

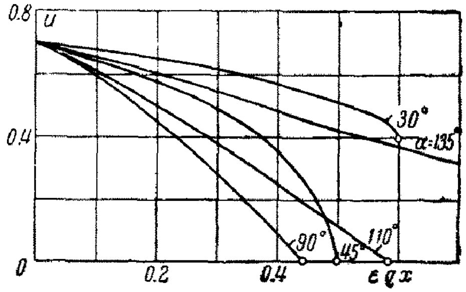

Fig. 2.

the flow will be the greatest (earliest separation). In Fig. 2, we give the function $u = u(\varepsilon qx)$ according to (2.4), and in Fig. 3, the function $p = p(\varepsilon qx)$ according to (2.3), for different values of the angle α and $\theta = 45^{\circ}$. All the curves, except the one for $\alpha = 135^{\circ}$, terminate at the cavitation point ($\alpha < 45^{\circ}$) or at the separation point ($\alpha > 45^{\circ}$). For $x = 0$, the quantities u and p for any α agree with the corresponding values calculated for the case of zero magnetic field: $u = \cos\theta$, $p = \sin^2\theta$.

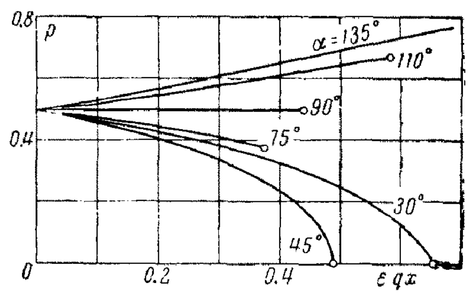

Fig. 3.

We now determine the form of the shock wave. Denoting in equation (1.8) the quantity $(\rho u)^{-1}$ by Φ (where $\Phi = \Phi(\psi + x\sin\theta)$ is a known function), carrying out the differentiation under the integral sign and then integrating, we get

$$v = -u\sin\theta\,[\Phi(\psi + x\sin\theta) - \Phi(x\sin\theta)] \tag{2.8}$$

Setting $\psi + x\sin\theta = 0$ in equation (2.8), we get an expression for v_0 on the shock wave. Substituting the resulting expression into (1.6) and using the values of Φ and u on the shock wave ($\Phi(0) = \varepsilon/\cos\theta$, $u(0) = \cos\theta$) and integrating (1.6) with respect to x, we have

$$\psi_0(x) = -x\sin\theta - \sin\theta\cos\theta\int_0^x \Phi(x\sin\theta)\,dx \tag{2.9}$$

$$\left(\int_0^x \Phi(x \sin\theta)\, dx = \int_0^x \frac{dx}{\rho_1 u_1}\right)$$

Here subscript 1 denotes quantities on the wall. From the first equation in (2.1), we have along the wall

$$\frac{dx}{\rho_1} = -\frac{du_1}{q \sin^2\alpha}$$

Using this equation, we integrate in (2.9) the expression $\Phi(x \sin\theta)$ and obtain

$$\psi_0(x) = -x\sin\theta + \frac{\sin\theta\cos\theta}{q\sin^2\alpha}\ln\frac{u_1(x)}{\cos\theta} \tag{2.10}$$

Finally, the equation of the shock may be written in the following form, according to (1.5):

$$y_0(x) = \frac{\sin\theta}{q\sin^2\alpha}\ln\frac{\cos\theta}{u_1(x)} \tag{2.11}$$

The solution obtained for $\alpha = \pi/2$ agrees with that [1] for the wedge. In particular, formulas (2.4), (2.6) and (2.11) become, respectively, formulas (2.9), (2.12) and (2.24) in [1]. (In formula (2.24) in [1] there is a misprint - a minus sign should appear in front of the right-hand side).

In conclusion, we determine the curvature of the shock at the vertex of the wedge. From (2.11) and the first equation of (2.1), we have

$$\frac{d^2y_0}{dx^2} = \frac{q\varepsilon^2\sin^2\alpha}{\cos\theta}(\tan\theta + 2\cot\theta + \cot\alpha) \tag{2.12}$$

From this formula, the following curious fact ensues.

For $\alpha = \alpha_0$, where

$$\cot\alpha_0 = -\tan\theta - 2\cot\theta \qquad (\pi/2 < \alpha_0 < \pi)$$

the shock curvature tends to zero.

For $\pi > \alpha > \alpha_0$ the shock curvature at the vertex of the wedge is negative, i.e. the shock wave in this case is convex against the flow.

In the neighborhood of the separation point, the shock curvature is positive - a qualitative picture of the flow for arbitrary α is like that given in [1] for $\alpha = \pi/2$. In other words, for $\pi > \alpha > \alpha_0$, the shock wave has an inflection point in the interval $0 < x < x_{**}$.

BIBLIOGRAPHY

1. Ladyzhenskii, M.D., Giperzvukovoe obtekanie tel v magnitnoi gidrodinamike (Hypersonic flow past a body in magnetohydrodynamics). *PMM* Vol. 23, No. 6, 1959.

2. Chernyi, G.G., *Techeniia gaza s bol'shoi sverkhzvukovoi skorost'iu* (*Introduction to Hypersonic Flow*). Fizmatgiz, 1959. (English translation, Academic Press, 1961).

3. Hayes, W.C. and Probstein, R.F., *Hypersonic Flow Theory*. Academic Press, N.Y., 1959.

4. Meyer, R.X., Magnetohydrodynamic hypersonic flow in the quasi-Newtonian approximation. *Rev. Mod. Phys.*, Vol. 32, No. 4, 1960.

5. Barthel, J.R. and Lykoidis, P.S., Magneto-fluid- dynamic problem of a shock wave attached to a cone. *Physics of Fluids*, Vol. 4, No. 12, 1961.

Translated by C.K.C.